

NOTE ON A SUM OF PRODUCTS WHICH INVOLVES
SYMMETRICALLY THE N^{TH} ROOTS OF 1.

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(1) Each of the products in question consists of n factors of the form

$$a_1 + a_2\theta + a_3\theta^2 + \dots + a_n\theta^{n-1},$$

where θ is an n th root of unity; and the sum includes n of these products, any two of which differ merely in that the n th root of unity appearing in the one is different from that which appears in the other. This is the same as saying that if ω be a primitive n th root of 1, the sum to be considered is

$$\sum_{s=1}^{s=n} \prod_{r=1}^{r=n} (a_{r1} + a_{r2}\omega^s + a_{r3}\omega^{2s} + \dots + a_{rn}\omega^{(n-1)s}).$$

(2) As a preliminary it is necessary to recall the notation of a class of functions studied in 1885 under the not very appropriate name of "bipartites," the word being adopted in consequence of it having been found that bilinear forms, called "bipartites" by Cayley, were viewable as functions of the kind in question when of the third degree.*

(3) The function of degree-order $2,n$ is

$$\frac{a_1, a_2, \dots, a_n}{b_1, b_2, \dots, b_n},$$

its equivalent in ordinary notation being

$$a_1b_1 + a_2b_2 + \dots + a_nb_n;$$

the function of degree-order $3,4$ is

$$\begin{array}{cccc|c} a_1 & a_2 & a_3 & a_4 & \\ \hline b_1 & b_2 & b_3 & b_4 & f_1 \\ c_1 & c_2 & c_3 & c_4 & f_2 \\ d_1 & d_2 & d_3 & d_4 & f_3 \\ e_1 & e_2 & e_3 & e_4 & f_4 \end{array}$$

which equals

$$\frac{a_1}{b_1} \frac{a_2}{b_2} \frac{a_3}{b_3} \frac{a_4}{b_4} \cdot f_1 + \frac{a_1}{c_1} \frac{a_2}{c_2} \frac{a_3}{c_3} \frac{a_4}{c_4} \cdot f_2 + \dots$$

and thus represents

$$a_1b_1f_1 + \dots + a_4e_4f_4,$$

the sum of sixteen terms got by taking each element of the square array

* 'Trans. R. Soc. Edinburgh,' xxxii, pp. 461-482.

and multiplying it by the outside elements which are in the same column or row with it: the function of degree-order 4,2 is

$$\begin{array}{c|c|c} a & b & \\ \hline c & d & g \ i \\ \hline e & f & h \ j \\ \hline & & k \ l \end{array} \quad \text{or} \quad \begin{array}{c|c|c} a & b & k \ l \\ \hline c & d & g \ i \\ \hline e & f & h \ j \\ \hline & & \end{array}$$

which equals

$$k \cdot \begin{array}{c|c|c} a & b & \\ \hline c & d & g \\ \hline e & f & h \end{array} + l \cdot \begin{array}{c|c|c} a & b & \\ \hline c & d & i \\ \hline e & f & j \end{array}$$

and therefore stands for

$$acgk + acil + aehk + aejl + bdgk + bdil + bfhk + bfjl:$$

and so on, the number of terms in the final expansion being n^{m-1} when the degree-order is m, n .

(4) The most interesting occurrence of the functions in analysis is as the elements of product-determinants: for example,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} = \begin{vmatrix} \begin{array}{c|c|c} a_1 & a_2 & a_3 \\ \hline p_1 & p_2 & p_3 \\ \hline b_1 & b_2 & b_3 \\ \hline c_1 & c_2 & c_3 \end{array} & \begin{array}{c|c|c} a_1 & a_2 & a_3 \\ \hline q_1 & q_2 & q_3 \\ \hline b_1 & b_2 & b_3 \\ \hline c_1 & c_2 & c_3 \end{array} & \begin{array}{c|c|c} a_1 & a_2 & a_3 \\ \hline r_1 & r_2 & r_3 \\ \hline b_1 & b_2 & b_3 \\ \hline c_1 & c_2 & c_3 \end{array} \end{vmatrix},$$

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \cdot \begin{vmatrix} h_1 & h_2 \\ k_1 & k_2 \end{vmatrix} \cdot \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix} = \begin{vmatrix} \begin{array}{c|c|c} a_1 & a_2 & \\ \hline h_1 & h_2 & m_1 \\ \hline k_1 & k_2 & m_2 \end{array} & \begin{array}{c|c|c} a_1 & a_2 & \\ \hline h_1 & h_2 & n_1 \\ \hline k_1 & k_2 & n_2 \end{array} \end{vmatrix},$$

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \cdot \begin{vmatrix} h_1 & h_2 \\ k_1 & k_2 \end{vmatrix} \cdot \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix} \cdot \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = \begin{vmatrix} \begin{array}{c|c|c|c} a_1 & a_2 & x_1 & x_2 \\ \hline h_1 & h_2 & m_1 & n_1 \\ \hline k_1 & k_2 & m_2 & n_2 \end{array} & \begin{array}{c|c|c|c} a_1 & a_2 & y_1 & y_2 \\ \hline h_1 & h_2 & m_1 & n_1 \\ \hline k_1 & k_2 & m_2 & n_2 \end{array} \end{vmatrix};$$

but several other distinct occurrences have been pointed out.*

(5) Let us now turn to the multiplying of

$$a_1 + a_2\omega + a_3\omega^2 + \dots + a_n\omega^{n-1}$$

by another expression of the like kind

$$b_1 + b_2\omega + b_3\omega^2 + \dots + b_n\omega^{n-1}.$$

* For example, 'Proc. Lond. Math. Soc.,' xvi, pp. 276-286.

It is readily manifest that in the product the term free of ω is

$$\frac{a_1, a_2, a_3, \dots, a_n}{b_1, b_n, b_{n-1}, \dots, b_2},$$

the cofactor of the first power of ω is

$$\frac{a_1, a_2, a_3, \dots, a_n}{b_2, b_1, b_n, \dots, b_3},$$

the cofactor of the second power is

$$\frac{a_1, a_2, a_3, \dots, a_n}{b_3, b_2, b_1, \dots, b_4},$$

and so forth. The total product thus is

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & \dots & a_n & \\ b_1 & b_n & b_{n-1} & \dots & b_2 & \\ b_2 & b_1 & b_n & \dots & b_3 & \\ b_3 & b_2 & b_1 & \dots & b_4 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_n & b_{n-1} & b_{n-2} & \dots & b_1 & \end{array} \begin{array}{c} 1 \\ \omega \\ \omega^2 \\ \vdots \\ \omega^{n-1} \end{array} \text{ or } \begin{array}{cccccc} 1 & \omega & \omega^2 & \dots & \omega^{n-1} & \\ b_1 & b_2 & b_3 & \dots & b_n & \\ b_n & b_1 & b_2 & \dots & b_{n-1} & \\ b_{n-1} & b_n & b_1 & \dots & b_{n-2} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_2 & b_3 & b_4 & \dots & b_1 & \end{array} \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{array}$$

where the important thing to note is that the rows of the square arrays are the circular permutations of

$$b_n, b_{n-1}, \dots, b_1 \text{ and } b_1, b_2, b_3, \dots, b_n$$

respectively—in other words, that the square arrays are the arrays of the circulant

$$C(b_1, b_n, b_{n-1}, \dots, b_2)$$

and its conjugate.

Of course, from the nature of the case, the a 's and b 's may be interchanged, and thus other two expressions for the result are obtainable.

(6) Confining ourselves, for convenience in writing, to the case where n is 4, let us now introduce a third factor

$$c_1 + c_2\omega + c_3\omega^2 + c_4\omega^3$$

the multiplicand being, as we have seen,

$$\frac{a_1 a_2 a_3 a_4}{b_1 b_4 b_3 b_2} + \frac{a_1 a_2 a_3 a_4}{b_2 b_1 b_4 b_3} \omega + \frac{a_1 a_2 a_3 a_4}{b_3 b_2 b_1 b_4} \omega^2 + \frac{a_1 a_2 a_3 a_4}{b_4 b_3 b_2 b_1} \omega^3.$$

In the new product the cofactors of $\omega^0, \omega^1, \omega^2, \omega^3$ are seen to be respectively

$$\begin{array}{cccc|c} a_1 & a_2 & a_3 & a_4 & \\ b_1 & b_4 & b_3 & b_2 & c_1 \\ b_2 & b_1 & b_4 & b_3 & c_4 \\ b_3 & b_2 & b_1 & b_4 & c_3 \\ b_4 & b_3 & b_2 & b_1 & c_2 \end{array}, \begin{array}{cccc|c} a_1 & a_2 & a_3 & a_4 & \\ & & & & c_2 \\ & & & & c_1 \\ & & & & c_4 \\ & & & & c_3 \end{array}, \begin{array}{cccc|c} a_1 & a_2 & a_3 & a_4 & \\ & & & & c_3 \\ & & & & c_2 \\ & & & & c_1 \\ & & & & c_4 \end{array}, \begin{array}{cccc|c} a_1 & a_2 & a_3 & a_4 & \\ & & & & c_4 \\ & & & & c_3 \\ & & & & c_2 \\ & & & & c_1 \end{array}$$

and therefore the whole product to be

$$\begin{array}{cccc|cccc} a_1 & a_2 & a_3 & a_4 & & & & \\ b_1 & b_4 & b_3 & b_2 & c_1 & c_2 & c_3 & c_4 \\ b_2 & b_1 & b_4 & b_3 & c_4 & c_1 & c_2 & c_3 \\ b_3 & b_2 & b_1 & b_4 & c_3 & c_4 & c_1 & c_2 \\ b_4 & b_3 & b_2 & b_1 & c_2 & c_3 & c_4 & c_1 \\ \hline & & & & 1 & \omega & \omega^2 & \omega^3 \end{array}.$$

The law of formation of the continued products is thus evident, and we can with confidence affirm the next result to be

$$\begin{aligned}
 & (a_1 + a_2\omega + a_3\omega^2 + a_4\omega^3)(\quad)(\quad)(d_1 + d_2\omega + d_3\omega^2 + d_4\omega^3) \\
 &= \frac{a_1 \ a_2 \ a_3 \ a_4}{b_1 \ b_4 \ b_3 \ b_2} \begin{vmatrix} c_1 & c_2 & c_3 & c_4 \\ b_2 & b_1 & b_4 & b_3 \\ b_3 & b_2 & b_1 & b_4 \\ b_4 & b_3 & b_2 & b_1 \end{vmatrix} \begin{vmatrix} d_1 & d_4 & d_3 & d_2 \\ d_2 & d_1 & d_4 & d_3 \\ d_3 & d_2 & d_1 & d_4 \\ d_4 & d_3 & d_2 & d_1 \end{vmatrix} \begin{vmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \end{vmatrix}.
 \end{aligned}$$

(7) Using Σ before any one of the products thus dealt with to indicate that ω is to be replaced by ω^2 , ω^3 , ω^4 , and the results to be added to the original, we obtain from §5

$$\begin{aligned}
 & \Sigma(a_1 + a_2\omega + a_3\omega^2 + a_4\omega^3)(b_1 + b_2\omega + b_3\omega^2 + b_4\omega^3) \\
 &= \frac{a_1 \ a_2 \ a_3 \ a_4}{b_1 \ b_4 \ b_3 \ b_2} \begin{vmatrix} 1 & +1 & +1 & +1 \\ \omega & +\omega^2 & +\omega^3 & +1 \\ \omega^2 & +1 & +\omega^2 & +1 \\ \omega^3 & +\omega^2 & +\omega & +1 \end{vmatrix} \\
 &= 4 \cdot \frac{a_1 \ a_2 \ a_3 \ a_4}{b_1 \ b_4 \ b_3 \ b_2},
 \end{aligned}$$

because of the vanishing of $1 + \omega + \omega^2 + \dots + \omega^{n-1}$. Similarly from §6 there are got

$$\begin{aligned}
 & \Sigma(a_1 + a_2\omega + a_3\omega^2 + a_4\omega^3)(\quad)(c_1 + c_2\omega + c_3\omega^2 + c_4\omega^3) \\
 &= 4 \cdot \frac{a_1 \ a_2 \ a_3 \ a_4}{b_1 \ b_4 \ b_3 \ b_2} \begin{vmatrix} c_1 \\ b_2 & b_1 & b_4 & b_3 \\ b_3 & b_2 & b_1 & b_4 \\ b_4 & b_3 & b_2 & b_1 \end{vmatrix} c_2,
 \end{aligned}$$

and $\Sigma(a_1 + a_2\omega + a_3\omega^2 + a_4\omega^3)(\quad)(\quad)(d_1 + d_2\omega + d_3\omega^2 + d_4\omega^3)$

$$= 4 \cdot \frac{a_1 \ a_2 \ a_3 \ a_4}{b_1 \ b_4 \ b_3 \ b_2} \begin{vmatrix} c_1 & c_2 & c_3 & c_4 \\ b_2 & b_1 & b_4 & b_3 \\ b_3 & b_2 & b_1 & b_4 \\ b_4 & b_3 & b_2 & b_1 \end{vmatrix} \begin{vmatrix} d_1 & d_4 & d_3 & d_2 \end{vmatrix}.$$

The last is the main result, the others being derivable from it by specialisation.

(8) As a consequence of the preceding, some interest attaches to determinants whose elements are of the type

$$a_1 + a_2\omega + a_3\omega^2 + \dots + a_n\omega^{n-1}.$$

Confining ourselves to the third order, merely from considerations of space, let us examine the determinant

$$\begin{vmatrix} a_1 + a_2\omega + a_3\omega^2 & b_1 + b_2\omega + b_3\omega^2 & c_1 + c_2\omega + c_3\omega^2 \\ d_1 + d_2\omega + d_3\omega^2 & e_1 + e_2\omega + e_3\omega^2 & f_1 + f_2\omega + f_3\omega^2 \\ g_1 + g_2\omega + g_3\omega^2 & h_1 + h_2\omega + h_3\omega^2 & k_1 + k_2\omega + k_3\omega^2 \end{vmatrix}.$$

By partitioning it into twenty-seven determinants with monomial elements it is readily seen to be expressible in the form

$$P + Q\omega + R\omega^2$$

where P, Q, R are each the sum of nine determinants, and in particular

$$\begin{aligned} P = & |147| + |258| + |369| \\ & + |159| + |267| + |348| \\ & + |168| + |249| + |357|, \end{aligned}$$

if $|m \ n \ r|$ stand for the determinant whose columns are the m^{th} , n^{th} , r^{th} of the 3-by-3² array

$$\begin{array}{ccc|ccc|ccc} a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 & e_1 & e_2 & e_3 & f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 & h_1 & h_2 & h_3 & k_1 & k_2 & k_3 \end{array}.$$

It follows therefore that, if we use Σ as above we have

$$P = \frac{1}{3} \sum \begin{vmatrix} a_1 + a_2\omega + a_3\omega^2 & b_1 + b_2\omega + b_3\omega^2 & c_1 + c_2\omega + c_3\omega^2 \\ d_1 + d_2\omega + d_3\omega^2 & e_1 + e_2\omega + e_3\omega^2 & f_1 + f_2\omega + f_3\omega^2 \\ g_1 + g_2\omega + g_3\omega^2 & h_1 + h_2\omega + h_3\omega^2 & k_1 + k_2\omega + k_3\omega^2 \end{vmatrix}.$$

Instead, however, of partitioning the determinant we may take the six terms of its ordinary expansion, and, these being of the type

$$(a_1 + a_2\omega + a_3\omega^2)(e_1 + e_2\omega + e_3\omega^2)(k_1 + k_2\omega + k_3\omega^2),$$

we can make use of our above theorem regarding such products, and thus arrive at the result

$$\begin{aligned} & \frac{1}{3} \sum \begin{vmatrix} a_1 + a_2\omega + a_3\omega^2 & b_1 + b_2\omega + b_3\omega^2 & c_1 + c_2\omega + c_3\omega^2 \\ d_1 + d_2\omega + d_3\omega^2 & e_1 + e_2\omega + e_3\omega^2 & f_1 + f_2\omega + f_3\omega^2 \\ g_1 + g_2\omega + g_3\omega^2 & h_1 + h_2\omega + h_3\omega^2 & k_1 + k_2\omega + k_3\omega^2 \end{vmatrix} \\ &= \frac{a_1 \ a_2 \ a_3}{e_1 \ e_3 \ e_2} \begin{vmatrix} k_1 \\ k_3 \\ k_2 \end{vmatrix} + \frac{b_1 \ b_2 \ b_3}{f_1 \ f_3 \ f_2} \begin{vmatrix} g_1 \\ g_3 \\ g_2 \end{vmatrix} + \frac{c_1 \ c_2 \ c_3}{d_1 \ d_3 \ d_2} \begin{vmatrix} h_1 \\ h_3 \\ h_2 \end{vmatrix} \\ &\quad - \frac{a_1 \ a_2 \ a_3}{f_1 \ f_3 \ f_2} \begin{vmatrix} h_1 \\ h_3 \\ h_2 \end{vmatrix} - \frac{b_1 \ b_2 \ b_3}{d_1 \ d_3 \ d_2} \begin{vmatrix} k_1 \\ k_3 \\ k_2 \end{vmatrix} - \frac{c_1 \ c_2 \ c_3}{e_1 \ e_3 \ e_2} \begin{vmatrix} g_1 \\ g_3 \\ g_2 \end{vmatrix}. \end{aligned}$$

Denoting this bipartite aggregate by T, we consequently have in regard to the above-mentioned 3-by-3² array the curious equality

$$P = T$$

connecting determinants and bipartites.

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